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The low Rossby number flow of a rotating fluid past a flat plate

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Summary

In a rotating fluid, the flow between two infinite plates, perpendicular to the rotation axis, is examined when a uniform stream is aligned with a finite flat plate, parallel to the rotation axis. Since the flow in this configuration is depth-independent the motion is analogous to that considered by Blasius in a non-rotating fluid. When the Rossby number Ro is much smaller than $E^{3/4}$, where E is the Ekman number, the equations are linear and the flow has been examined by Hocking [5]. However, when $Ro \gg E^{3/4}$ inertial effects are important in the $E^{1/4}$-layer and the boundary-layer equations are non-linear. For Ro of order $E^{1/2}$ the boundary-layer flow is calculated numerically and very close to both the leading and trailing edges of the plate the flow is identical to that in the non-rotating case. Goldstein expansions are calculated at both points and the singularity at the trailing edge is examined using triple-deck theory. This demonstrates that for Ro of order $E^{1/2}$ the $E^{1/4}$-layer exhibits behaviour similar to that of a classical boundary layer.

1. Introduction

In this paper inertial effects in $E^{1/4}$-layers in a rapidly rotating fluid are examined for a simple flat-plate configuration, analogous to that considered by Blasius in a non-rotating fluid, to establish a correspondence with results from classical boundary-layer theory. This strengthens the relationship between these two types of boundary layers which has been demonstrated in other configurations (Walker and Stewartson [1], Walker and Stewartson [2], Page [3]) when the Rossby number Ro is $O(E^{1/2})$, where E is the Ekman number. Furthermore, in this problem it is shown that triple-deck theory can be applied to inertially modified $E^{1/4}$-layers over the range of Rossby numbers $E^{3/4} \ll Ro \ll E^{1/3}$.

The configuration examined, relative to a rotating frame, is two-dimensional flow past a flat plate of finite length, aligned with a uniform stream. Such motion can be produced in a rapidly rotating frame by forcing a uniform flow between two parallel plates, perpendicular to the rotation axis, past a finite plate which is aligned with both the stream and the axis of rotation. The two-dimensionality of the resulting motion is assured by the Taylor-Proudman theorem (Greenspan [4]).

For $Ro = 0$ this configuration, which is illustrated in Fig. 1, is considered in a paper by Hocking [5] which shows that the structure consists of Stewartson $E^{1/3}$ and $E^{1/4}$-layers, with the $E^{1/4}$-layer flow being independent of distance along the plate except within a distance $O(E^{1/4})$ from the leading and trailing edges. The solution in these edge regions is calculated by the Wiener-Hopf technique and, in particular, it is shown to decay

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algebraically away from the plate. This solution is an accurate representation for all Rossby numbers for which $\mathrm{Ro} \ll \mathrm{E}^{3/4}$.

When the Rossby number is $O(\mathrm{E}^{3/4})$ the inertial terms in the governing equations are significant in both of the edge regions and consequently the leading- and trailing-edge solutions are no longer identical, as they are for $\mathrm{Ro} \ll \mathrm{E}^{3/4}$. At both these edges it is necessary to solve a non-linear elliptic equation of fourth order for the stream function, but this has not been attempted here.

For $\mathrm{Ro} \gg \mathrm{E}^{3/4}$, which is the main parameter regime examined in this study, the extent in the streamwise direction of both of the edge regions is altered from $O(\mathrm{E}^{1/4})$ to $O(\mathrm{Ro}/\mathrm{E}^{1/2})$. In these regions the governing equation is parabolic and similar, although not identical, to the classical boundary-layer equation. It is essentially the same equation as that derived by Walker and Stewartson [1] for a different external flow. In particular, the parabolic character of the equation indicates that both the leading- and trailing-edge regions extend only downstream, and not upstream from the edges.

Close to the leading edge the solution is similar to the Blasius flow, with inertial and viscous forces dominating Coriolis effects, but as the distance from the leading edge increases the solution tends to the simple distance-independent flow seen when $\mathrm{Ro} = 0$. At the trailing edge the solution changes from this form back to a uniform flow in a similar manner to that described by the classical solution of Goldstein, although the wake now decays exponentially in the streamwise direction, rather than algebraically. Once Ro is $O(\mathrm{E}^{1/2})$, however, the leading-edge region extends beyond the trailing edge of the plate and the flow is still distance-dependent at the trailing edge. Although this case is not considered explicitly in this paper, the general principle of the calculation remains the same.

For Rossby numbers much larger than $O(\mathrm{E}^{1/2})$ the thickness of the boundary layer decreases, proportional to $(\mathrm{Ro}/\mathrm{E}^{1/2})^{-1/2}$, and both the leading- and trailing-edge flows are, to lowest order, identical to the corresponding non-rotating flows. The rotation is therefore not of primary importance to the flow in this case, even though the Rossby number is still smaller than unity.

In classical boundary-layer theory the trailing edge solution, as derived by Goldstein [6], is singular at the end of the plate with an infinite normal velocity at that point. This singularity has been resolved by the application of triple-deck theory (Stewartson [7], Messiter [8]) which introduces a smaller length-scale region at that point. A similar structure is shown, in this paper, to be appropriate in the rotating case although the scales of the regions are functions of the two independent parameters Ro and E, rather than the single parameter $\mathrm{Re} = \mathrm{Ro}/\mathrm{E}$.

2. Governing equations

Consider an imcompressible fluid, of density ρ^* and kinematic viscosity ν^*, contained between two infinite plates, a distance d^* apart, all rotating about an axis which is perpendicular to the plates at an angular velocity $\Omega^*\hat{\boldsymbol{k}}$. A uniform flow of speed U^* is forced in the fluid, relative to the rotating frame, so that it is aligned with a thin plate of length l^* which spans the gap between the two infinite plates. The configuration is illustrated in Fig. 1. In terms of these dimensional quantities, three dimensionless parameters Ro, E and l can be defined, namely

$$\mathrm{Ro} = U^*/\Omega^* d^*, \qquad \mathrm{E} = \nu^*/\Omega^* d^{*2}, \qquad l = l^*/d^*, \tag{2.1}$$

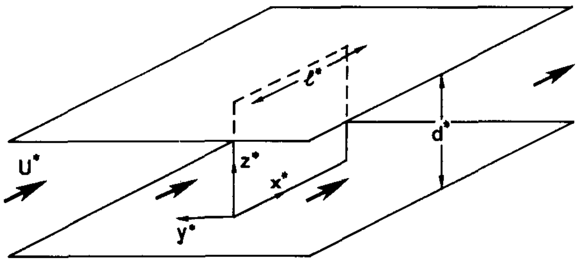

Fig. 1. Geometrical configuration studied, relative to a frame rotating about the z^*-axis at angular velocity Ω^*.

and these are the Rossby number, Ekman number and scaled length, respectively. In this paper both Ro and E will be considered to be small and l is taken to be of order unity.

In Cartesian coordinates the dimensionless position and velocity, relative to the rotating frame, are defined as

$$\boldsymbol{x} = (x, y, z) = \boldsymbol{x}^*/d^*, \qquad \boldsymbol{u} = (u, v, w) = \boldsymbol{u}^*/U^*, \tag{2.2}$$

where the z-axis is aligned with the axis of rotation. The origin is defined so that the plate is at $0 \leqslant x \leqslant l$, $y = 0$, $0 \leqslant z \leqslant 1$ and the velocity at large distances is $\boldsymbol{u} = (1, 0, 0)$. The equations of motion in these coordinates are then

$$\mathrm{Ro}(\boldsymbol{u} \cdot \nabla_1)\boldsymbol{u} + 2(\hat{\boldsymbol{k}} \times \boldsymbol{u}) = -\nabla_1 P + \mathrm{E}\nabla_1^2 \boldsymbol{u}, \tag{2.3}$$

$$\nabla_1 \cdot \boldsymbol{u} = 0, \tag{2.4}$$

where P is the reduced pressure, defined by

$$P = \left[p^* - \tfrac{1}{2}\rho^*(\Omega^* \hat{\boldsymbol{k}} \times \boldsymbol{x}^*)^2\right]/\rho^* U^* \Omega^* d^* \tag{2.5}$$

and ∇_1, ∇_1^2 are the three-dimensional gradient and Laplacian operators. The boundary conditions on the motion are $\boldsymbol{u} = \boldsymbol{0}$ on all solid surfaces and $\boldsymbol{u} \to (1, 0, 0)$ as $x^2 + y^2 \to \infty$ for $0 < z < 1$.

For $\mathrm{Ro} \ll 1$ and $\mathrm{E} \ll 1$ the momentum equation (2.3) is, to lowest order, the geostrophic equation

$$2(\hat{\boldsymbol{k}} \times \boldsymbol{u}) = -\nabla_1 P \tag{2.6}$$

and therefore, in the bulk of the fluid, the motion is depth-independent and two-dimensional. Hence the velocities u, v are functions of (x, y) only and, from (2.4), w is zero to lowest order. To solve for these quantities it is necessary to evaluate higher order terms in (2.3), which can most easily be done by eliminating P from the (x, y) components, leading to the equation

$$\mathrm{Ro}\left[u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right] = (2 + \mathrm{Ro}\,\zeta)\frac{\partial w}{\partial z} + \mathrm{E}\nabla^2\zeta, \tag{2.7}$$

for the z-component of vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2.8}$$

In (2.7) the symbol ∇^2 represents the two-dimensional Laplacian since ζ is independent of z to leading order, and consequently $\partial w/\partial z$ is also independent of z. This latter quantity can therefore by evaluated from the Ekman conditions on $z = 0$, 1 (Greenspan [4]) to be

$$\frac{\partial w}{\partial z} = -\mathrm{E}^{1/2}\zeta. \tag{2.9}$$

Neglecting the term Roζ, which can be shown to be small *a posteriori* provided Ro $\ll \mathrm{E}^{1/3}$, and defining the two parameters

$$\lambda = \mathrm{Ro}/2\mathrm{E}^{1/2}, \qquad \delta = \left(\tfrac{1}{2}\mathrm{E}^{1/2}\right)^{1/2}, \tag{2.10}$$

the vorticity equation becomes

$$\lambda\left[u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y}\right] = -\zeta + \delta^2\left[\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right]. \tag{2.11}$$

This equation, along with the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.12}$$

is sufficient to determine the lowest order flow in the majority of the fluid, including the $\mathrm{E}^{1/4}$-layers. It does not, however, determine the $\mathrm{E}^{1/3}$-layer flow on the plate, but this will not be considered in this paper since it is clear that the appropriate conditions to apply at the inner edge of the $\mathrm{E}^{1/4}$-layer are $u = v = 0$.

Away from the plate, equation (2.11) implies that $\zeta = 0$ and therefore the unique solution is

$$\boldsymbol{u} = (1, 0, 0). \tag{2.13}$$

Since this solution does not satisfy the no-slip condition $u = 0$ on the plate, there is a boundary layer on $y = 0$, $0 \leqslant x \leqslant l$ and it is this layer which is the interesting feature in the flow. In the following section it will be discussed in detail for various magnitudes of Rossby number.

3. Parameter regimes

In the limit of zero Rossby number, so that the non-linear inertial terms may be neglected, the flow in the boundary layer for the configuration described in Section 2 is described by Hocking [5]. In that paper it is shown that, away from the edges of the plate at $x = 0$, l the boundary-layer flow is independent of x and given by

$$u = 1 - \exp(-\bar{y}), \tag{3.1}$$

where $\bar{y} = y/\delta \geqslant 0$. With this scaling the vorticity equation (2.11) reduces to the linear equation

$$-\zeta + \delta^2 \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial \bar{y}^2} = 0, \tag{3.2}$$

where the second term, which represents viscous diffusion in the x-direction, is zero except near $x = 0$, l. It follows, from (3.2), that in these edge regions the x-scale is $O(\delta)$, the same as the y-scale, and that both terms on the right-hand side of (2.8) are $O(\delta^{-1})$. Therefore, when posed in terms of a suitably-defined stream function, the problem is fourth order and elliptic. In Hocking [5], this has been solved using the Wiener-Hopf technique.

As the Rossby number is increased, the convective terms in (2.11) become increasingly important. Also, unlike the case $\lambda = 0$, the flows near the two edges of the plate are no longer identical for non-zero Rossby numbers since the equations are no longer symmetric about $x = \frac{1}{2}l$. The size of the edge regions does, however, remain $O(\delta)$ and the x-independent solution (3.1) still describes the flow away from $x = 0$, l. Once λ is $O(\delta)$ so that Ro is $O(\mathrm{E}^{3/4})$, the inertial terms are equal in magnitude to all the terms in (3.2) and the flow at the two ends can only be found by solving a nonlinear equation. The associated numerical problem has not been attempted in this study.

As the Rossby number is increased through $O(\mathrm{E}^{3/4})$ the convection terms in (2.11) play an increasingly important role in the two edge regions, and therefore when $\mathrm{Ro} \gg \mathrm{E}^{3/4}$ they can be expected to balance with the two terms $-\zeta$ and $\delta^2 \partial^2 \zeta / \partial y^2$ in (2.11), which are important in the $\mathrm{E}^{1/4}$-layer on the plate. From this it follows that the x-lengthscale of the regions at $x = 0$, l increases from $O(\delta)$ when Ro is $O(E^{3/4})$ to $O(\lambda)$ when $\mathrm{Ro} \gg \mathrm{E}^{3/4}$, with viscous diffusion in the x-direction insignificant in the latter case. In addition, the vorticity to leading order is given simply by $\zeta = -\partial u/\partial y$ so that (2.11) can be integrated to give the boundary-layer type of equation

$$\lambda \left[u \frac{\partial u}{\partial x} + \bar{v} \frac{\partial u}{\partial \bar{y}} \right] = 1 - u + \frac{\partial^2 u}{\partial \bar{y}^2}, \tag{3.3}$$

where $\bar{v} = v/\delta$, and

$$\frac{\partial u}{\partial x} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0. \tag{3.4}$$

This equation is similar to that derived by Walker and Stewartson [1] for flow around a circular cylinder, where it is shown to resemble the classical boundary-layer equation in character. A similar equation has also been studied by Page [3] in a study of separation of the $\mathrm{E}^{1/4}$-layer in the presence of a decelerating external flow when Ro is $O(\mathrm{E}^{1/2})$.

For $\mathrm{E}^{3/4} \ll \mathrm{Ro} \ll \mathrm{E}^{1/2}$ the leading and trailing-edge regions occupy only a small proportion, of order $\mathrm{Ro}/\mathrm{E}^{1/2}$, of the length of the plate, with the boundary-layer solution (3.1) valid along the remainder of it. However, once Ro is $O(\mathrm{E}^{1/2})$ the leading-edge region has a length scale of the same order as the length of the plate. Therefore, the x-independent solution (3.1) is not necessarily attained within the range $0 \leqslant x \leqslant l$ so that the flow at the trailing edge can be affected by the presence of the leading edge. A more complete study of the flow when the Rossby number is $O(\mathrm{E}^{1/2})$ is presented in Section 4.

Increasing the Rossby number beyond $O(\mathrm{E}^{1/2})$ it can be seen that the structure of the flow changes further. In particular, in Page [9] it is seen that the flow over all of the plate is, to leading order, the Blasius solution of classical boundary-layer theory. Furthermore, the thickness of the boundary layer decreases to $O(\delta\lambda^{-1/2})$, or $O((E/Ro)^{1/2})$, for $\lambda \gg 1$. This latter property places an upper bound on the range of Rossby numbers for which the theory is valid because the term Ro ζ, neglected in (2.11), becomes order unity once Ro is $O(E^{1/3})$. At this value $\delta\lambda^{-1/2}$ is $O(E^{1/3})$, so that the inertially-modified $E^{1/4}$-layer has become the same thickness as the depth-dependent $E^{1/3}$-layer. In addition, the equations for the flow in the Ekman layers above the boundary layer are no longer linear when Ro is $O(E^{1/3})$, so that non-linear effects in those layers must be taken into account, perhaps in the manner of Bennetts and Hocking [10].

In the next section a more detailed examination will be made of the most significant parameter range, that is when $E^{3/4} \ll Ro \ll E^{1/3}$.

4. Leading- and trailing-edge flows

In Section 3 it was seen that for $\delta \ll \lambda \ll E^{-1/6}$ the flow in the boundary layer on the plate satisfies the non-linear parabolic equations (3.3) and (3.4). The solution of these equations near the leading and trailing edges will now be examined in detail and it will be seen that for Rossby numbers in this range the flow is not dissimilar to that in the classical boundary-layer problem of flow past a finite flat plate.

Close to the leading edge of the plate the inertial and viscous terms in (3.3) dominate the flow and therefore to lowest order the flow is the same as for the non-rotating case where Coriolis forces are absent. Therefore similarity variables are introduced close to $x = 0$ so that

$$\xi = (x/\lambda)^{1/2}, \qquad \eta = \tfrac{1}{2}\bar{y}(x/\lambda)^{-1/2} \tag{4.1}$$

and the velocity u is written in the form

$$u = \frac{1}{2}\frac{\partial f}{\partial \eta}(\xi, \eta), \tag{4.2}$$

where f satisfies the equation

$$\frac{\partial^3 f}{\partial \eta^3} + f\frac{\partial^2 f}{\partial \eta^2} + \xi\left[\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \eta \partial \xi}\right] = 8\xi^2\left[\frac{1}{2}\frac{\partial f}{\partial \eta} - 1\right]. \tag{4.3}$$

Close to the leading edge f can be expanded in the form

$$f(\xi, \eta) = \sum_{n=0}^{\infty} \xi^{2n} f_n(\eta) \tag{4.4}$$

and the leading order term f_0 satisfies the Blasius equation

$$\frac{d^3 f_0}{d\eta^3} + f_0\frac{d^2 f_0}{d\eta^2} = 0. \tag{4.5}$$

Downstream from the leading edge the flow is expected to decay to the x-independent solution (3.1) and to examine how this is attained the similarity equation (4.1) was integrated numerically, starting from the initial solution f_0 at $\xi = 0$. The skin friction

$$\tau = \tfrac{1}{4}\xi^{-1}\frac{\partial^2 f}{\partial \eta^2}(\xi, 0) \tag{4.6}$$

and the displacement thickness

$$\delta^* = \lim_{\eta \to \infty} \xi(f - 2\eta) \tag{4.7}$$

calculated from the numerical solution are shown in Fig. 2. This illustrates how the flow develops from a Blasius form when $\xi \ll 1$ to (3.1), for which $\tau = \delta^* = 1$, when $\xi \gg 1$. The numerical results also indicate that the flow decays exponentially in x, a feature which can also be confirmed theoretically (Page [9]), and that τ is within 1% of its asymptotic value when $\xi = 1.0$. Therefore, the effect of the leading edge is, in practice, limited to a distance of about λ from $x = 0$, so for $\lambda < l$ the flow at the trailing edge is given by (3.1). For $\lambda > l$, however, the velocity profile at the trailing edge must be calculated numerically and, although this is not considered explicitly, the results which follow apply in principle.

For the non-rotating case the flow in the wake of the trailing edge was calculated as a series solution by Goldstein [6] and, for the same reasons as those outlined for the leading edge, the solution close to the edge can be expected to be the same in this case, apart from a change in scale. Goldstein's expansion is in the similarity variables

$$\tilde{\xi} = \left(\frac{x - l}{\lambda}\right)^{1/3}, \qquad \tilde{\eta} = \tfrac{1}{3}\bar{y}\left(\frac{x - l}{\lambda}\right)^{-1/3} \tag{4.8}$$

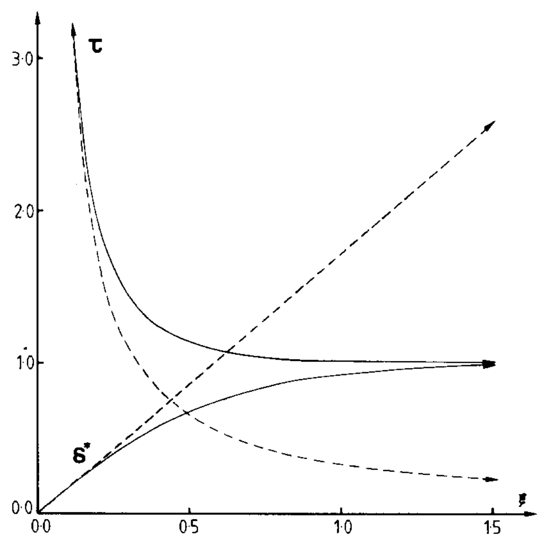

Fig. 2. Skin friction τ and displacement thickness δ^* near the leading edge, where $\xi = (x/\lambda)^{1/2}$. The corresponding results for a non-rotating fluid are shown by a broken line.

with the streamwise velocity in the form

$$u = \tfrac{1}{3}\tilde{\xi}\frac{\partial g}{\partial \tilde{\eta}}(\tilde{\xi}, \tilde{\eta}). \tag{4.9}$$

The resulting equation for g is then

$$\frac{\partial^3 g}{\partial \tilde{\eta}^3} + 2g\frac{\partial^2 g}{\partial \tilde{\eta}^2} + \tilde{\xi}\left[\frac{\partial g}{\partial \tilde{\xi}}\frac{\partial^2 g}{\partial \tilde{\eta}^2} - \frac{\partial g}{\partial \tilde{\eta}}\frac{\partial^2 g}{\partial \tilde{\eta}\partial \tilde{\xi}}\right] - \left(\frac{\partial g}{\partial \tilde{\eta}}\right)^2 = 27\tilde{\xi}\left[\tfrac{1}{3}\tilde{\xi}\frac{\partial g}{\partial \tilde{\eta}} - 1\right] \tag{4.10}$$

which for $\tilde{\xi} \gg 1$ gives $g \sim 3\tilde{\eta}/\tilde{\xi}$ and $u \to 1$. For $\tilde{\xi} \ll 1$ an expansion can be carried out in the form

$$g(\tilde{\xi}, \tilde{\eta}) = \sum_{n=0}^{\infty} \tilde{\xi}^n g_n(\tilde{\eta}) \tag{4.11}$$

and the leading-order term satisfies

$$\frac{d^3 g_0}{d\tilde{\eta}^3} + 2g_0\frac{d^2 g_0}{d\tilde{\eta}^2} - \left(\frac{dg_0}{d\tilde{\eta}}\right)^2 = 0, \tag{4.12}$$

as in Goldstein's calculations for the non-rotating case. In particular this implies that there is a singularity in the flow at $\tilde{\xi} = 0$ since for $\bar{y} \ll 1$ the normal velocity $\bar{v}$ is discontinuous at that point with

$$\bar{v} \sim -\frac{0.297}{\lambda\tilde{\xi}^2}\bar{y} \qquad \text{as} \quad \tilde{\xi} \to 0. \tag{4.13}$$

This singularity is discussed further in Section 5.

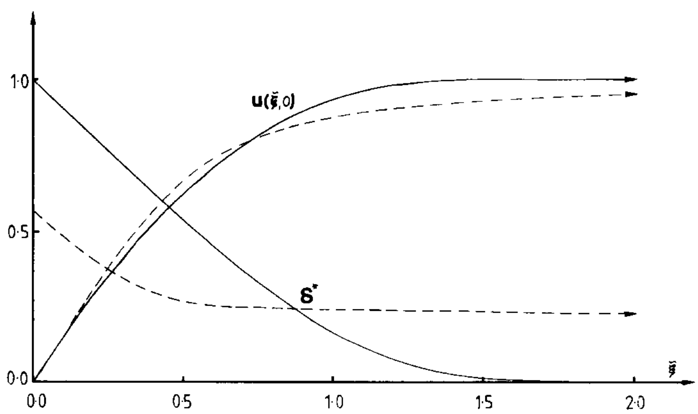

Fig. 3. Centre-line velocity $u(\tilde{\xi}, 0)$ and displacement thickness δ^* near the trailing edge, where $\tilde{\xi} = [(x - l)/\lambda]^{1/3}$. The corresponding results for a non-rotating fluid are shown by a broken line.

The Goldstein outer expansion can be calculated readily with the profile (3.1) at $\xi = 0$ and it reduces to the form (Page [9])

$$u = 1 - \exp\left(-\bar{y} + \delta^*(\tilde{\xi}) - 1\right) \tag{4.14}$$

where δ^* is the displacement thickness. This expression simplifies the numerical calculation of the inner flow, given by (4.10), since the boundary conditions for that flow can be calculated from (4.14) rather than directly from the oncoming and potential flows at $\tilde{\xi} = 0$ and $\bar{y} = \infty$. Using the condition

$$\frac{1}{9}\frac{\partial^2 g}{\partial \tilde{\eta}^2} \sim 1 - \tfrac{1}{3}\tilde{\xi}\frac{\partial g}{\partial \tilde{\eta}} \qquad \text{as} \qquad \tilde{\eta} \to \infty \tag{4.15}$$

the inner flow was calculated numerically and the results are shown in Fig. 3. The displacement thickness δ^* was calculated from g by evaluating

$$\delta^* \sim 1 + 3\tilde{\xi}\tilde{\eta} - \tilde{\xi}^2 g - \tfrac{1}{3}\tilde{\xi}\frac{\partial g}{\partial \eta} \tag{4.16}$$

for $\tilde{\eta}$ large. Also shown in Fig. 3 are the equivalent calculations for a non-rotating flow, taken from Goldstein [6] with the scaling for $\tilde{\xi}$ and $\tilde{\eta}$ chosen so that the leading-order terms in g are the same. Of particular interest is the exponential decay of the wake for the rotating case, and the decay to zero of the displacement thickness, whereas δ^* tends to a constant value in Goldstein's calculations. Similarly, the momentum thickness tends to zero, rather than a constant, as $\tilde{\xi} \to \infty$ which implies that there is, far downstream, no momentum defect in the wake behind the plate. This interesting result is due to the effect of the Coriolis forces exactly balancing the loss in momentum in the fluid due to the shear stress on the plate.

5. Triple-deck theory

In Section 4 the flow near the trailing edge has been derived and it has been noted that for a wide range of values of λ the solution is singular at $x = l$. This is most apparent in the normal velocity component $\bar{v}$ in the outer solution, since

$$\bar{v} = -\frac{0.297}{\lambda^{1/3}(x-l)^{2/3}}\left[1 - \exp(-\bar{y})\right] \qquad \text{as} \qquad x \to l+, \tag{5.1}$$

and it follows that there must be a small region at the trailing edge where this singularity is resolved. A similar singularity is present in the non-rotating case which has been examined by Stewartson [7] and Messiter [8] using triple-deck theory, and therefore a similar strategy is employed here.

The x-scale of the region where this singularity is resolved must be smaller than $O(\lambda)$ and therefore a scaled coordinate X is introduced where

$$(x - l) = \lambda\varepsilon^3 X \tag{5.2}$$

and ε is a small, as yet unknown, parameter. From Stewartson [7] it follows that there are three decks to this region and that the largest, the upper deck, has a y-scale of $\lambda\varepsilon^3$ and consists of potential flow. The other sub-regions are the main deck, with the same thickness as the boundary layer, and the lower deck which is thinner than $O(\delta)$. In particular, the lower deck must match onto the Goldstein inner solution as $X \to \infty$ so that

$$u \sim \varepsilon X^{1/3} g_0'(\bar{y}/(3\varepsilon X^{1/3})) \tag{5.3}$$

from which it follows that the thickness of the deck is $O(\varepsilon\delta)$. Consequently, the scaled variables

$$Z = \frac{y}{\varepsilon\delta}, \qquad U = \frac{u}{\varepsilon}, \qquad V = \frac{\lambda\varepsilon^2 v}{\delta}, \tag{5.4}$$

are introduced and, from (2.11), it follows that

$$U\frac{\partial^2 U}{\partial X \partial Z} + V\frac{\partial^2 U}{\partial Z^2} = \frac{\partial^3 U}{\partial Z^3}, \qquad \frac{\partial U}{\partial X} + \frac{\partial V}{\partial Z} = 0 \tag{5.5}$$

or, integrating the vorticity equation once,

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Z} = -\frac{\mathrm{d}P}{\mathrm{d}X} + \frac{\partial^2 U}{\partial Z^2}, \tag{5.6}$$

where $\mathrm{d}P/\mathrm{d}X$ is an unknown function of X. From (5.6) it is apparent that the Coriolis term has no leading order contribution in this layer, being of relative order ε^2/λ. The boundary conditions on the flow are

$$\left.\begin{array}{l} U = V = 0 \\ \dfrac{\partial U}{\partial Z} = V = 0 \end{array}\right\} \quad \text{on} \quad Z = 0, \quad \left\{\begin{array}{l} X < 0 \\ X > 0 \end{array}\right. \tag{5.7}$$

with the initial condition $U \to Z$ as $X \to -\infty$ and a matching condition with the main deck as $Z \to \infty$. In the main deck the Coriolis terms are also of relative order ε^2/λ and therefore, as in Stewartson [7], the solution is of the form

$$u = u_0(\bar{y}) + \varepsilon A(x) u_0'(\bar{y}) + \dots \tag{5.8}$$

where $u_0 = [1 - \exp(-\bar{y})]$ and A is another unknown function of X. Matching this with the lower deck gives

$$U \sim Z + A(X) \qquad \text{as} \qquad Z \to \infty. \tag{5.9}$$

It now remains to determine ε and to establish a relation between A and $\mathrm{d}P/\mathrm{d}x$ which will close the lower deck problem. Both of these goals are achieved by matching with the upper deck which gives

$$\varepsilon^4 = \frac{\delta}{\lambda} \qquad \text{and} \qquad P = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A'(x_1)}{x - x_1} \mathrm{d}x_1. \tag{5.10}$$

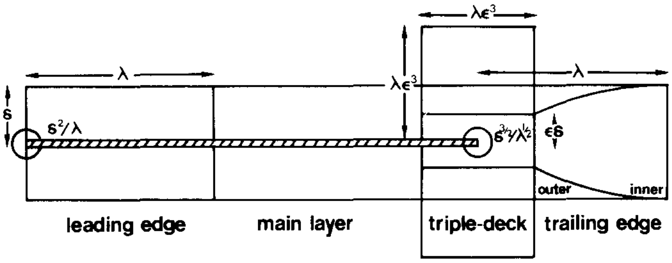

Fig. 4. Sketch of the important flow regions for Rossby numbers of $O(E^{1/2})$ (not to scale).

Therefore in the scaled variables the lower deck problem is identical to that derived by Stewartson and solved by Veldman and Van de Vooren [11], Jobe and Burggraf [12] and Chow and Melnik [13], so the solution for the non-rotating case can be applied directly to this case.

An interesting feature of the triple deck is its dependence upon two independent parameters λ and δ, rather than on a single parameter. Therefore as λ is decreased the relative sizes of the decks change so that they all merge together when $\lambda = O(\delta)$, reflecting the structure described for that case in Section 3.

Finally, in Fig. 4 the whole plate is illustrated with the various regions of the flow and their sizes (not to scale). In addition to those described here there are small regions at the leading and trailing edges of radius $O(\delta^2/\lambda)$ and $O(\delta^{3/2}/\lambda^{1/2})$, respectively, which are equivalent to the corresponding regions in the non-rotating case which have been described by Carrier and Lin [14] and Stewartson [15].

6. Conclusion

The foregoing analysis, although it is for a specific and idealised geometry, can be extended readily to more general geometries and situations because of the direct analogy which has been established with existing work in two-dimensional classical boundary-layer theory. The literature on the latter is extensive but for triple deck theory in particular the results of Smith [16], Brown and Stewartson [17] and Riley and Stewartson [18] can all be applied to rotating-fluid situations once the suitable modifications have been made. Further details of these applications are given in Page [9].

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